Explore

TRIANGLES

The 180° rule will be repeatedly used and is the sole commonality among all the different triangles the students will learn. You may, therefore, want to write it on the board for the duration of the class.

Angle Family

Review angle terminology and properties such as acute, obtuse, right, etc. This will be the basis upon which triangles can be categorized.

Right Triangles

Right triangles are prevalent throughout ACT Math. Emphasize how mastering the characteristics of the different triangles can help start the process of solving many Geometry problems.

Answer: $\angle B$ is the right angle. The right angle will always be the biggest angle because the angles in a triangle must always add up to 180o. If one angle is already 90o, then no other angle in a triangle can be equal to – or greater than – 90o.

Explore

TRIANGLES

There are many different types of triangles, and you will soon learn how and why they got their name. The sides and angles of triangles collectively tell stories about the shapes, and understanding the relationships between these sides and angles will be crucial to manipulating triangles in more complex ACT Geometry problems.

The most important fact to remember is that the angles in a triangle will *always* add up to a total of 180°. This means that, given any two angles in a triangle, you will be able to calculate the measure of the third angle. How the angles add up to 180° is what separates triangles from each other. You will learn in this chapter how to identify triangles based on the angle measurements and their corresponding sides.

At the end of this lesson, you will be able to

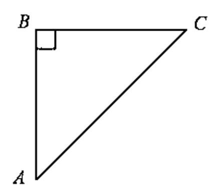
a. name and categorize triangles into different families
b. use formulas to determine the lengths of the sides in certain triangles
c. identify special triangles that appear quite frequently in ACT Geometry
d. correlate special triangles to other geometric shapes and figures
e. be able to recognize relationships among the sides in special right triangles
f. calculate triangle properties such as perimeter and area

Angle Families

The following triangles are characterized by the greatest angles within them.

Right Triangles

Triangles are characterized by the measure of their *greatest* angle. Triangles whose greatest angle is 90°, or a **right angle**, are called **right triangles**. The ACT loves to test right triangles, so we'll talk more about these shortly.



In the above right triangle, which angle is the right angle? Is the right angle always going to be the biggest angle in a right triangle? Why or why not?

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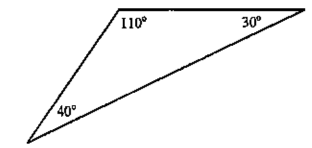
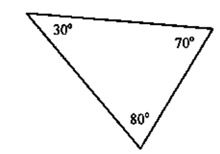
Obtuse Triangles

If a triangle's greatest angle is an obtuse angle (greater than 90°), then the triangle is called an **obtuse triangle**. The below image is an example of an obtuse triangle. Which angle is the obtuse angle?



Acute Triangles

If all of the angles in a triangle are acute (smaller than 90°), then the triangle is called an **acute triangle**. The greatest angle in an acute triangle is smaller than 90°. The below image is an example of an acute triangle.



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Obtuse Triangles

Answer: The 110° angle is the obtuse angle since it is greater than 90°.

Side Families

It is important to note that the following categories are not exclusive of the aforementioned triangles. Several of these categories will overlap, so students could correctly name triangles in a variety of ways. Listing the following categories on the board will allow you to easily reference them throughout the duration of this chapter and the subsequent exercises.

Equilateral Triangles

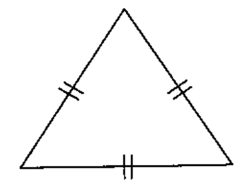
Answer: Each angle in an equilateral triangle is 60°.

Side Families

Now let's examine some triangles that are categorized by their sides.

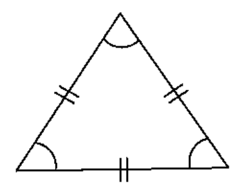
Equilateral Triangles

Triangles with three equal sides are called **equilateral triangles** because the root *equi* means "equal" and *lateral* means "side." We symbolize a triangle as being equilateral by drawing two little lines (like an equal sign) across each side of the triangle. The below image is an example of an equilateral triangle with the symbols on each side.



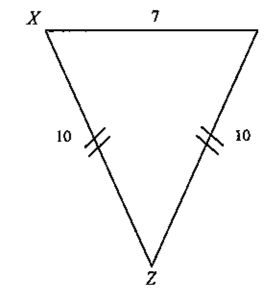
Equilateral triangles are special not only because all of their sides are equal but also because all of their angles are equal. Since that is the case, can you calculate how many degrees each angle in an equilateral triangle is?

We can symbolize that the angles are all equal to each other by drawing little curves in each angle. These marks say that the angles in an equilateral triangle are **congruent**.



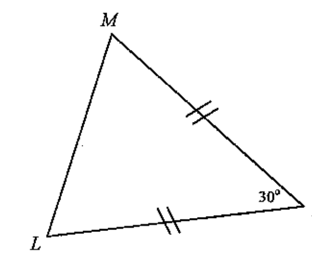
Isosceles Triangles

Another type of triangle that has equal sides is the **isosceles triangle.** An isosceles triangle has at least two equal sides (also called **congruent** sides). The below image is an example of an isosceles triangle.



Because sides $\overline{XZ}$ and $\overline{YZ}$ are congruent, we can symbolize this as $\overline{XZ} \cong \overline{YZ}$. Furthermore, the angles in an isosceles triangle that are opposite, or across from, the congruent sides are also congruent; thus, in the above image, $\angle X \cong \angle Y$. Therefore, we now have two ways in which we can categorize a triangle as isosceles: (1) two equal sides or (2) two equal angles.

Let's try some exercises now that we've learned about a couple different families of triangles.



Isosceles Triangles

1. Answer: $\overline{MN}$ and $\overline{LN}$ are congruent to each other. $\angle M \cong \angle L$. $\angle M = 75°$. You may want to present this problem on the board with all appropriate labels so that students can easily follow along.
2. Answer: ΔLMN could also be categorized as an acute triangle.

Right Triangles

The Pythagorean Theorem should also be written on the board. Students should not only know *how* to use it but also *when* to use it. Oftentimes, students will incorrectly apply the Pythagorean Theorem to a triangle that does not have a right angle.

Work this problem on the board, step by step, to reinforce the importance of writing out every step when solving a problem.

Students will often simply apply the special triplet to a triangle without ensuring that the triangle is a right triangle or that the longest side is applied to the hypotenuse. This is, therefore, a very important rule to emphasize.

1. ΔLMN is isosceles. What sides are congruent to each other? Which angles are congruent to each other? How many degrees is $\angle M$?

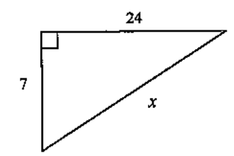
2. How else could you categorize ΔLMN? Is it an acute, obtuse, or right triangle?

Right Triangles

In right triangles, if the lengths for two sides are given, a special formula can be used to calculate the third side: the **Pythagorean Theorem**. The Pythagorean Theorem is defined as follows:

$$a^2 + b^2 = c^2$$

In this formula, *a* and *b* are the legs (sides around the right angle) of the right triangle, and *c* is the longest side of the right triangle, which is called the **hypotenuse**. The Pythagorean Theorem is extremely powerful because whenever we know the lengths of two sides of any right triangle, we can use it to solve for the length of the final side! The below image illustrates how to use the Pythagorean Theorem.



$$a^2 + b^2 = c^2$$

$$7^2 + 24^2 = x^2$$

$$49 + 576 = x^2$$

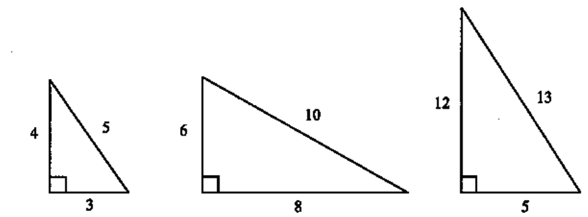
$$625 = x^2$$

$$25 = x$$

After squaring each of the lengths of the sides of the right triangle, the sum of those squared values is equal to the square of the hypotenuse, so the hypotenuse ($\overline{AC}$) is equal to 25.

This 7-24-25 right triangle combination of numbers can also be called a Pythagorean triplet. If you simply memorized the sequence of these numbers, you wouldn't have to work the entire Pythagorean Theorem out. How many of them should you memorize? Just three combinations!

In ACT Geometry, the following Pythagorean triplets are the most common ones: **3-4-5**, **6-8-10** (which is just 3-4-5 doubled), and **5-12-13**. If a right triangle (and make sure that it is a *right triangle*!) has legs of lengths 3 and 4, then using the Pythagorean Theorem won't be necessary to identify the length of the hypotenuse.



If you're ever unsure about which Pythagorean triplet to use (or just don't remember them), you can always use the Pythagorean Theorem to get your answer.

Sneak Peek

Answer: A.

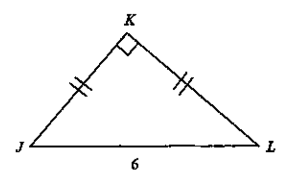
Special Right Triangles

The 30-60-90 and 45-45-90 triangles will not be provided on the ACT, so students should learn them

You can remind students here that the smallest sides are always opposite the smallest angles, and the biggest sides are always opposite the biggest angles.

This is an interesting point to reinforce because this association can be extremely beneficial in solving more complex ACT Geometry problems.

Sneak Peek



8. What is the area of ΔJKL?

A. 9

B. 12

C. 18

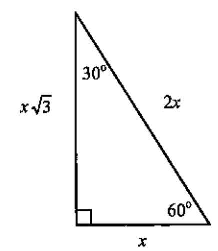
D. 36

E. 72

Special Right Triangles

30-60-90 Triangles

30-60-90 triangles all have angles that measure 30°, 60°, and 90°. The lengths of their sides are always in the same **ratio** (or **proportion**) to each other. That means that if you know just one side's length, you can multiply or divide that length to get the two other sides.



The side across from the 30° angle is always the smallest side because it is across from the smallest angle. We'll call that side x in the above figure.

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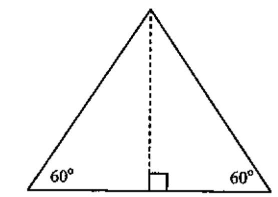
The side across from the 60° angle is always x times $\sqrt{3}$. We'll call that $x\sqrt{3}$.

The side across from the 90° angle, the hypotenuse, is always x times 2. We'll call that side $2x$. That's easy to remember because 90° is the biggest angle, and $2x$ is the longest side! We'll call that side $2x$.

A fun fact about the 30-60-90 triangle is that it is exactly half of an **equilateral** triangle.



If you draw a vertical line (the height) from the top point to the base of an equilateral triangle, splitting it into two congruent triangles, you create two 30-60-90 triangles. You can use those 30-60-90 triangles to help you find the height if you need it.

Isosceles Right Triangles

Isosceles right triangles always have angles that measure 45°, 45°, and 90°. With these triangles, as with the 30-60-90 triangles, the lengths of their sides are always in the same ratio to each other. That means that if you know just one side's length, you can find the length of the other two sides. This triangle type is a little easier because there are two **congruent** sides – the two legs of the triangle.

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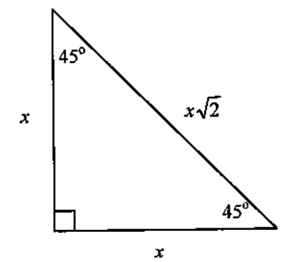
Isosceles Right Triangles

Emphasizing the correlation between the special right triangles and other geometric shapes will give students more tools to solve Geometry problems. Putting this relationship on the board – and the correlation between 30-60-90 triangles and equilateral triangles – will visually reinforce this helpful teaching point.



In the above isosceles right triangle, let's call each of the two legs x. The side across from the right angle, or the hypotenuse, is always just x times $\sqrt{2}$. If you put two 45-45-90 triangles together, you get a square. If you draw a diagonal line through a square and split it into two congruent triangles, you create two isosceles right triangles.

If you ever get confused about which triangles uses $\sqrt{3}$ and which ones uses $\sqrt{2}$, remember that the triangle with three different angles (30-60-90) uses $\sqrt{3}$, and the triangle that has just two different angle measures (45-45-90) uses $\sqrt{2}$.

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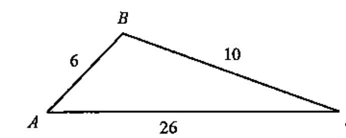
Perimeter

The perimeter is the distance around the outside of a shape, so, in this chapter, we'll be discussing the distances around triangles. To find the perimeter, you don't need to use any particular formula. All you have to do is add up all of the sides.

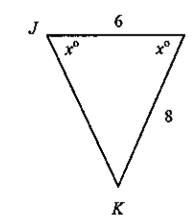


Since the three sides of this triangle are 6, 10, and 26, we can add them.

$$6 + 10 + 26 = 42$$

Therefore, the perimeter of ΔABC is 42.

Sometimes, you'll have to use other information you know, such as information about angles or area, to get the lengths of all three sides. Check out this example:



Since ΔJKL has two equal angles, there must be two equal sides, but which two? Triangle angles are always related to the side across from them. Therefore, the third side of ΔJKL must have a length of 8 because it is across from the other x.

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Perimeter

ACT Geometry can be extremely tricky sometimes when asking for perimeters. When questions involve multiple shapes, it can be helpful to outline the shape whose perimeter students must calculate so that they don't fall for trap answers.

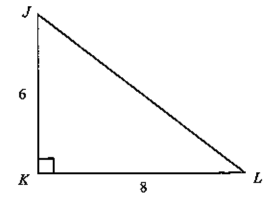
Area

You may need to walk the students through the process of splitting the 10 into 5 and 5 and then applying the 30-60-90 ratio.

Area

Area is the amount of space inside a shape. There is a special formula that is used to find the area A of a triangle:

$A = \frac{1}{2}bh$, where b is the length of the **base**, or bottom, of the triangle, and h is the height of the triangle, or how tall it is. The base and the height MUST be **perpendicular** to each other; therefore, the point at which the height and base meets is a 90° angle. Remember the symbol for perpendicular? In the below triangle, we can say that $\overline{JK} \perp \overline{KL}$.



We see that the above triangle has a height of 6 and a base of 8. Putting that into the formula, we get the following:

$$A = \frac{1}{2}bh$$

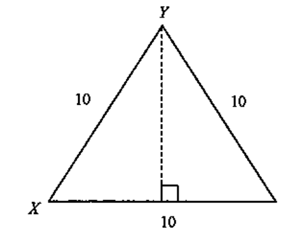
$$A = \frac{1}{2}(8)(6)$$

$$A = \frac{1}{2}(48)$$

$$A = 24$$

Thus, the area of ΔJKL is equal to 24.

For an equilateral triangle, we still need to use the same area formula, but the height will take a little work to solve. Take the below triangle:



Remember that an equilateral triangle is simply made up of two 30-60-90 triangles? Knowing this, we can use the known relationships among the sides of a 30-60-90 triangle to label the height of ΔXYZ as $5\sqrt{3}$ since it is across from the 60° angle. Applying the formula for the area of a triangle, we get the following:

$$A = \frac{1}{2}bh$$

$$A = \frac{1}{2}(10)(5\sqrt{3})$$

$$A = \frac{1}{2}(50\sqrt{3})$$

$$A = 25\sqrt{3}$$

Thus, the area of ΔXYZ is $25\sqrt{3}$.